Partial Default*

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Abstract

In the data sovereign defaults are always partial and vary in their intensity and duration, and the intensity of defaults covaries with interest rate spreads, debt, and economic activity. This paper develops a theory for sovereign default that replicates these properties, which are absent in standard sovereign default theory. Partial default is a flexible way to raise funds as the sovereign chooses the intensity and the duration of default. Partial default is also costly because it amplifies debt crises as the defaulted debt accumulates, increasing interest rate spreads and making new borrowing more expensive. This theory is capable in rationalizing the large heterogeneity in partial default, its comovements with economic outcomes, and rising debt during default episodes. In our theory, as in the data, large defaults are associated with higher interest rate spreads, higher debt levels, deeper recessions, and longer default episodes.

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1 Introduction

Understanding the forces that lead to sovereign debt crises and defaults is central for emerging markets as default risk is a main driver for fluctuations in capital flows and economic activity. Sovereigns in these countries frequently miss payments on their debt, but the fraction of payments missed is usually small, and these episodes are often short-lived. Governments on average default about one-third of the time, on one-third of payments due, and about 40% of the episodes last less than 2 years. During these *partial defaults*, sovereigns continue to pay some of the debt, borrow at higher than normal rates, and accumulate the defaulted debt as arrears. The standard theory of sovereign default, as in the influential framework in Eaton and Gersovitz (1981), assumes that default is complete and is followed by a period of exclusion with no continued debt payment, borrowing, or accumulation of arrears.[1] In this paper, we propose a theory of partial default that is more in accord with the evidence.

A central idea in our theory is that partial default is an alternative way to effectively borrow and inter-temporally transfer resources. As with standard borrowing, partially defaulting raises current resources and increases future liabilities, as the defaulted debt accumulates. Unlike standard borrowing, however, partial default does not have the acquiescence of the lenders, because it is repaid with a discount and is associated with future resources costs. A main implication of our theory is that partial default is a powerful amplifying force for large and prolonged debt crises. A country that misses a small fraction of its payments is in much worse shape going forward because it will experience fast-rising debt as the defaulted payments accumulate and the new borrowing occurs at high interest rates. Partial default leads to more default. This theory is capable in rationalizing the large heterogeneity in partial default, its comovements with economic outcomes, and rising debt during default episodes. In our theory, as in the data, large defaults are associated with higher interest rate spreads, higher debt levels, deeper recessions, and longer default episodes.

Our analysis puts at center-stage partial default, which we define as the fraction in arrears of current debt payments due by sovereigns. We document the properties of partial default using data for 38 emerging markets since 1970. We find that sovereigns default often, with varying intensity and duration, missing their debt payments about one-third of the time. During these events, often only a small fraction of the payment is missed; in about half of these events, less than

[1] For a robust review of this literature, see the recent handbook chapters of Aguiar and Amador (2014) and Aguiar et al. (2016).

30% of the promised payments are missed. Default episodes vary in duration, with many lasting less than 2 years but some on occasion lasting much longer, over 30 years. Debt-to-output ratios feature a hump-shaped pattern during default episodes, and these episodes are not associated with a reduction in debt. As default episodes start, debt-to-output rises from 35% to 40%. In the middle of the episode, debt is higher than at the beginning and equal to 47%; debt falls to 36% after the default episode. We also find that partial default is systematically correlated with other outcomes of the debt crisis. During small defaults, sovereign governments miss on average 3% of payments, interest rate spreads are about 7%, debt-to-output ratios are about 37%, and output is at trend. During large defaults, in contrast, governments miss about 82% of their payments, face interest rate spreads of about 15%, debt-to-output ratios are 63%, and output is about 6% below trend. Large defaults are also associated with longer default episodes.

Our framework consists of a sovereign government in a small open economy that borrows long-term bonds, can choose to partially default on its debt payments, and faces a stochastic stream of income. Partial default is a flexible yet expensive way to raise funds. Partial default is flexible because the government can choose when to start the default episode, the intensity of partial default every period, and when to end the default episode. The defaulted debt is accumulated and repaid in the future at a discount, yet it is costly because default induces future resource costs that depend on the intensity of the default. The government can also raise resources by borrowing at interest rates that compensate for potential default losses. Borrowing is always possible, even during default episodes. Expected default losses, however, are much more elevated during default episodes, which increases interest rates and can deter borrowing altogether. This framework leads to varying debt haircuts and maturity extensions that depend on the endogenous default episode length and the intensity of the default.

The sovereign effectively faces a portfolio decision to intertemporally transfer resources and smooth consumption because it chooses how much to borrow and partially default every period. We characterize theoretically the trade-offs for the sovereign of these two choices. The marginal gains from borrowing are the increases in consumption by the bond price net of reduction in price due to higher default, while its marginal costs consist of the future coupon payments evaluated at future prices which encode the discount from not repaying in all states. The marginal gains are capped by a Laffer Curve for borrowing that arises in sovereign default models due to default risk. The marginal benefits of partially defaulting increase linearly with the amount defaulted on and is not subject to a Laffer Curve, but is capped by the total level of debt. The marginal costs from partially defaulting are also the future payments from the accumulation of the defaulted debt with a discount, also evaluated at future prices plus the future resource costs arising from default.

The portfolio of borrowing and partially defaulting results from evaluating these marginal gains and costs. In equilibrium, when income is high and debt is low, borrowing is preferred to partial default because these are times when borrowing rates are low. As income falls and debt rises, borrowing rates rise and a mix of borrowing and partial default is optimal. When debt rises further and income falls more the benefits from defaulting are larger leading default is complete and the costs from borrowing is highest borrowing rates are extremely high. Default increases borrowing rates because defaulted debt piles up as arrears. The bond price schedule reflects these default decisions and shapes the dynamics of the model.

We estimate our model by targeting moments that summarize the empirical distribution of partial default and the behavior of interest rate spreads, debt, and output in emerging markets. We estimate the parameters of the default cost function as well as the discount from partial default and show that our over-identified model matches the target moments well and delivers the empirical distribution of debt and partial default, with many small defaults.

The estimated model contains additional implications, consistent with the data, for the correlations of partial default with other variables, as well as the length of default episodes and debt haircuts and maturity extensions from default episodes. In the model and the data, small defaults tend to be shorter and are associated with lower interest rate spreads, lower levels of debt, and smaller recessions. Large partial defaults in the model and data are longer and associated with much higher interest rate spreads, higher levels of debt, and larger recessions. The model and the data feature default episodes lasting many years, yet also feature many short defaults. Our model also matches the data in that it delivers a hump-shaped pattern for debt to output during episodes. During default episodes, debt continues to rise in the model, and default episodes do not result in a reduction of the debt burden. Default episodes in our model also result in sizable debt haircuts and maturity extensions with magnitudes similar to those in the data.

To assess the mechanisms behind these quantitative results, we analyze impulse responses to income shocks. When income declines, the bond price schedule tightens, making it more costly to roll over the debt. To alleviate the consumption decline, the government not only increases borrowing at higher interest rates but also partially defaults. The rise in borrowing and partial default increases the future debt and creates the dynamic amplification for shocks. As debt remains elevated, interest rates and partial default remain persistently high, even as the shock dissipates. Recessions in our model have long-lasting effects on the functioning of international financial markets as debt levels rise from increased borrowing and the accumulation of the defaulted debt.

We also analyze the dynamics of default episodes directly. The severity of the debt crisis and default episode depends on the interaction between the magnitude and persistence of the recession and the accumulation of debt from past borrowing and partial default. Default episodes generally start with a small partial default that occurs after a moderate downturn when debt is high enough. These small defaults are resolved quickly if the recession is temporary. When the recession is larger and more persistent, however, the small partial defaults amplify the debt crisis by inducing a rapid increase in debt at increasingly high interest rates. Small partial defaults coupled with persistent recessions are the culprit of prolonged debt crises. The episode ends when output recovers enough to repay the accumulated debt. Larger debt crises require stronger output recoveries for resolution of default as they feature larger accumulated debt from past borrowing and partial defaults.

Default episodes are associated with debt haircuts and maturity extensions that depend on the discount in future repayment achieved with partial default and also depend on the length of the episode. Longer episodes have larger haircuts because the default discount is compounded with repeated partial defaults. The model features maturity extensions because defaults occur on payments due, which are inherently short term, and the repayment occurs with the long-term bonds. The path and intensity of partial default also shape maturity extensions because the overall maturity of the defaulted debt depends on whether larger defaults occur early or late during the episode.

We perform two counterfactual experiments that resemble recent discussions around resolution mechanisms for sovereign defaults. In the first counterfactual, we eliminate the possibility of borrowing during default episodes and show that in our model, this policy is similar to adding more stringent pari passu clauses to the bond contracts. Borrowing during default episodes lead to differential haircuts to lenders after default episodes because bonds issued later in the episode experience fewer periods with partial default compared to bonds issued earlier, and such differential treatment violates Pari-Pasu clauses. In the second counterfactual, we introduce stochastic discounts from default and show that a debt reprofiling policy can be interpreted as situations in which the realizations of the discounts are small. Debt reprofiling, a policy recently supported by the International Monetary Fund, consists of relying more heavily on maturity extensions and less on debt relief for resolving defaults. Our counterfactual suggests that pari passu clauses lead to lower default frequency, shorter default episodes, and smaller debt haircuts from defaults. Reprofiling policies reduce haircuts, but in contrast to pari passu clauses, aggravate default episodes by generating higher default frequencies, longer default episodes, and higher debt-to-output ratios.

Related Literature This paper introduces the study of partial default in the literature on sovereign debt and embraces the lack of commitment inherent in sovereign contracts. In addressing new aspects of the evidence, it departs in essential ways from work in the tradition of Eaton and Gersovitz (1981). In the standard short-term debt model of Aguiar and Gopinath (2006) or Arellano (2008), default can only be on the total amount of debt, is followed by a period of exclusion, and ends in a full discharge of debt. These features are also true of extensions of the model to long-term debt, including Chatterjee and Eyigungor (2012) or Hatchondo, Martinez, and Sosa-Padilla (2016), where default is not only on the current debt service but also on all the future coupons. In these papers, a default episode, spanning the moment of default and the specified moment of redemption, precludes any continuation of debt repayments on any coupons, further borrowing, or accumulation of arrears. During default episodes, the sanction of complete exclusion from financial markets is assumed and rationalized by assumed commitment from lenders to collude in not lending to the sovereign. By design, such theory cannot accommodate the observable implications for the partial default episodes that we measure and study here.

An active recent research agenda extends this baseline model to add less blunt resolutions for defaults. Yue (2010) and Benjamin and Wright (2013) develop models for debt renegotiations for resolving defaults. Yue (2010) models renegotiation via Nash bargaining, while Benjamin and Wright (2013) model renegotiation with a dynamic alternating offers framework. Recent work in Mihalache (2017) and Dvorkin et al. (2018) extend the renegotiation framework to long-term bonds and endogenous maturity choice. Here, as in our work, the length of the default episode, the debt haircut, and the maturity extensions after default are endogenous. Default episodes in these frameworks end with haircuts that depend on bargaining forces, and the delay in the resolution arises via a delay in settlement. Moreover, in Mihalache (2017) and Dvorkin et al. (2018), default episodes also end with an extension in the maturity of the debt. Nevertheless, just as in the earlier papers, in these papers, default episodes are an impasse state. In contrast to our work and actual data, while in default the borrower does not continue debt repayment, or borrow, and the defaulted debt does not accumulate.

Our work is also related to the literature on private defaultable debt and personal bankruptcy. As in the literature of sovereign debt, the majority of the work, as in Chatterjee et al. (2007) and Livshits, MacGee, and Tertilt (2007), has focused on full defaults and private bankruptcy. In this context the assumption that default is a discrete action upon which debts are discharged coincide with much of bankruptcy law, where debt is formally discharged. Recently, however, analyzing partial defaults is gaining more attention because defaults outside formal bankruptcy procedures are substantial, as documented by Dawsey and Ausubel (2004). In Mateos-Planas and

Seccia (2014), households default partially on their debts giving rise to incomplete consumption insurance in an environment with a complete set of securities. **?** model foreclosure as a process that takes a long time between payments are stopped and when the house is lost. In this context, non-payment acts as a forced form of financing by the lenders as in our model. Still, borrowers do not choose how much to pay back, just whether to repay and exit the foreclosure process or to continue the process towards its legal resolution.

2 The Empirical Properties of Sovereign Defaults

In this section we use panel data for emerging markets and document the properties of sovereign defaults. We first document that governments default with varying intensity and duration and that many defaults are small and short. We also document that default is systematically correlated with outcomes of the debt crisis: large defaults are associated with higher interest rate spreads, larger debt levels, deeper recessions, and longer default episodes. Finally, we document that default episodes do not lead to a decrease in debt.

2.1 Data

We use a panel dataset for 38 emerging markets from 1970 to 2010. The sample of countries consists of those from the J.P. Morgan Emerging Markets Bond Index (EMBI+). We use public debt statistics from the World Development Indicators (WDI) and public bond spreads from the Global Financial Database. From the WDI, we construct measures for *debt* defined as the ratio of total government debt public and publicly guaranteed (PPG) to output, and *output* defined as the detrended log of real output, detrended with a linear country-specific trend. From the Global Financial Database, we get annual series for the EMBI+ government bond spreads, which we use as our measure of *spreads*.

We also use information on debt in arrears to construct the panel series for default intensity, which we label as *partial default*. For each year and country, we define *arrears* as the sum of interest and principal in arrears (PPG). We define *debt due* as the sum of debt service (PPG) and arrears. We define partial default as the ratio of the arrears to the debt due for each year and country. This variable is our measure of default intensity:

$$\text{Partial Default} = \frac{\text{Arrears}}{\text{Arrears} + \text{Debt Service}}. \quad (1)$$

Our definition of partial default essentially measures the fraction of payments missed. With the time series measure of partial default for many countries, we also study the duration of default episodes. We measure the duration of default episodes, which we label as *default episode length*, as the number of consecutive years that a country has positive values for partial default. The appendix contains the list of countries and all the variables in more detail.

2.2 Empirical Findings

We document the distributions of partial default and default episode length and their comovement with interest rate spreads, debt-to-output ratios, and output.

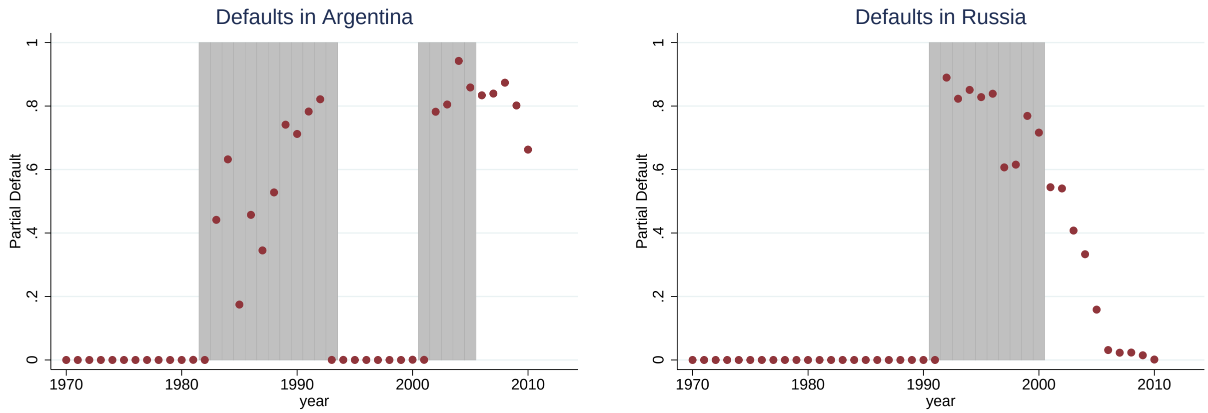
We start by describing some time series of partial default for two emerging market countries with a history of sovereign default: Argentina and Russia. In Figure 1 we plot the time series of partial default from 1970 to 2010 for these countries. The figure shows that default varies in intensity and is always partial, ranging from small levels of less than 10%, as was the case in the late 2000s for Russia, to high levels of more than 90%, as in the case of Argentina in the early 2000s. In terms of default episode length, Argentina experienced two episodes with lengths equal to 10 and 9 years, and Russia experienced one episode of a length equal to 20 years.

Although our partial default measure is distinct from the popular Standard and Poor's (S&P) binary definition of default, it correlates with it.[2] The shaded areas in Figure 1 are the years S&P classifies Argentina and Russia as being under default. The figure shows that having positive partial default is correlated with having the S&P default flag.

We now study the properties of partial default for the 38 emerging markets. Sovereigns partially default often, and defaults vary in intensity and duration. The frequency of positive partial default in the panel dataset is 35%. The varying intensity of partial default across years and countries is illustrated in Figure 2(a). Here we plot the histogram of partial default (conditional on positive partial default) for panel data, the year×country series of the 38 emerging markets. The histogram shows that countries partially default at varying degrees covering the full range. Sovereigns often

[2] S&P records a sovereign as being in default if it has failed to meet any payments on the due date and removes the default flag after it settles a payment with creditors.

default on a small amount of debt due; about 37% of the time, partial default is less than 20%.

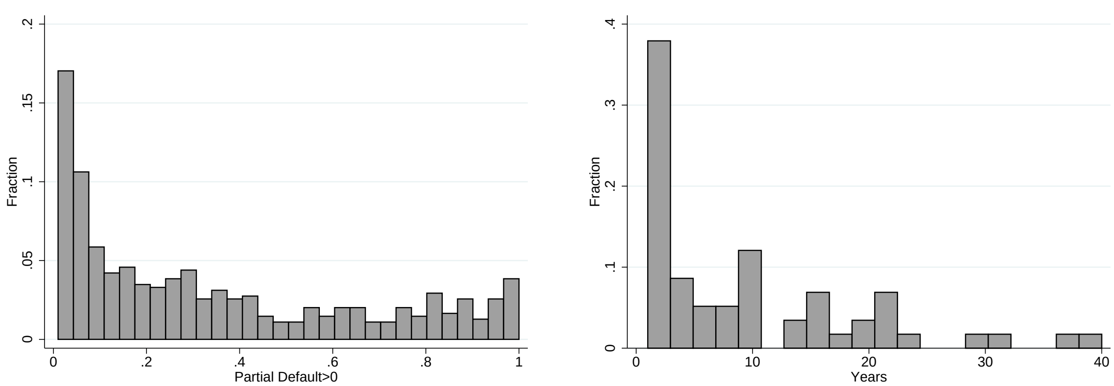


(a) Argentina

(b) Russia

Figure 1: Time Series for Partial Default

Default episodes also vary in duration. In Figure 2(b) we plot the histogram of default episode length for the 58 default episodes in the dataset. Most of the default episodes are short-lived; almost 40% of the episodes last less than 2 years. The histogram has a long right tail, as few episodes last more than 30 years. The distribution of the default episode length in our dataset is similar to the one documented in **?**.



(a) Partial Default

(b) Default Episode Length

Figure 2: Partial Default and Default Episode Length

In Table 1 we summarize the distributions of partial default and default episode length. Emerging markets often have positive partial default, the average intensity of the default is modest, yet the variation is large. The mean partial default conditional on positive partial default is also 35% with a standard deviation of 16%. The mean length of the default episode is equal to 9 years, but a large fraction of the defaults are short. As the table shows, the fraction of default episodes that last less than 2 years is 38%.

The table also reports the dynamics of debt to output during default episodes. Debt to output feature on average a hump-shaped pattern during default episodes, and default episodes do not lead to reductions in debt. In the period before the episode starts, the ratio of debt to output is 35%. The default episode starts with a higher debt-to-output ratio of 40%. During the default episode, the debt continues to rise and is equal to 47% in the middle of the episode.[3] Debt decreases toward the end of the episode and equals 36% in the period after the episode ends. Our findings are consistent with those in Benjamin and Wright (2013) that find using a different dataset, that debt is not smaller following the end of the default episode than it was prior to the episode.

Table 1: Partial Default and Default Episodes

Partial Default $\mid > 0$	
Frequency	35
Mean	35
Standard deviation	16
Default Episodes	
Episode length (years)	9
Fraction of short episodes ($\leq$ 2 years)	38
Debt During Episode	
Before episode	35
Beginning of episode	40
Middle of episode	47
After episode	36

Next we document the comovement of partial default with spreads, debt to output, output, and episode length. Table 2 shows that partial default is positively correlated with spreads and debt to output, with correlations equal to 25% and 34%, and negatively correlated with output and equal to -12%. The correlation between partial default and default episode length is computed across the 58 default episodes where partial default is the mean value for the episode. Partial default is positively correlated with episode length and equal to 55%; longer episodes are more complete.

To illustrate the magnitude of the comovement of partial default with interest rate spreads, debt to output, and output, we divide the panel dataset into four bins based on the levels of partial default and report for each bin the mean of the variables of interest. We report the means

[3] We define the middle of the episode as the total length of the episode divided by 2 rounded to the nearest integer.

Table 2: Correlations with Partial Default

Spreads	25
Debt to output	34
Output	-12
Episode length	55

of partial default, spreads, debt-to-output ratios and output across the partial default bins in Table 3. The no default bin consists of the observations with zero partial default. We partition the observations with positive partial default into three groups. The small partial default bin contains the observations below the 25th percentile; the medium partial default bin contains the observations between the 25th and 75th percentiles; and the large partial default bin contains the observations above the 75th percentile.

By construction, the means of partial default across the groups display a large variation in default. Small defaults on average have only 3% of their payments in default, medium defaults have 27% of their payments in default, and large defaults have an average of 82% of their payments in default.

Table 3: Partial Default, Spreads, Debt, and Output

	No default	Partial default > 0		
Means (%)		Small	Medium	Large
Partial default	0	3	27	82
Spreads	4	7	7	15
Debt to output	26	37	49	63
Output	1	0	-3	-6

Table 3 shows that spreads, debt to output, and output have sizable differences as partial default varies. Spreads in periods of no default are on average 4%. During small and medium defaults, spreads rise modestly to an average of 7%. During large defaults, however, spreads more than double and are on average 15%. Debt to output in periods of no default is on average 26% and is higher in periods when sovereigns partially default. Debt to output rises monotonically to 37%, 49%, and 63% in periods of small, medium, and large defaults. The higher debt-to-output ratios during partial default run counter to the benchmark narrative that governments default to reduce their debt burden. The increase in debt during default occurs because defaulted debt is accumulated as debt in arrears and because governments continue to borrow new loans while partial default is positive. Finally, output in periods of no default is on average 1% above trend.

Output deteriorates as default rises and reaches -6% below trend during large defaults.[4]

3 The Model

We consider a dynamic model of borrowing with partial default. A small open economy has a stochastic endowment stream, borrows long-term debt, and can choose to partially default on the debt due. The defaulted debt accumulates over time until it is repaid at a discount. Partial default imposes resource costs on the economy that are increasing in the intensity of the default. Borrowing rates reflect default risk and compensate creditors for expected losses.

The duration of the default episode is endogenous as it depends on when the economy chooses to start defaulting on the debt and when to exit by repaying the defaulted debt. Partial default is persistent because as debt accumulates after default, the debt crises perdure and are amplified. Partial default also worsens borrowing terms, as default leads to fewer resources and larger debt due in the future. The duration of the default episode determines the haircuts and maturity extensions that creditors get from default episodes.

3.1 The sovereign borrower

Each period the sovereign receives a stochastic endowment z_t that follows a Markov process with transition probabilities $\pi(z_{t+1}, z_t)$. The sovereign discounts time at rate β and maximizes expected utility over consumption sequences, c_t, with preferences given by

$$E\left\{\sum_{t=0}^{\infty}\beta^t u(c_t)\right\}. \tag{2}$$

The actual income of the sovereign, y_t, is the endowment z_t net of any costs associated with default.

The sovereign trades long-term defaultable bonds with international creditors. Long-term bonds are perpetuity contracts with coupon payments that decay geometrically at rate δ, as in Hatchondo

[4]These comovements are related to results in Cruces and Trebesch (2013) and Tomz and Wright (2013). Cruces and Trebesch (2013) find that spreads are higher following default episodes with large haircuts during renegotiations. Tomz and Wright (2013) document that output tends to be lower during default episodes.

and Martinez (2009). The perpetuity contract specifies a price q_t and a face value b_t such that the economy receives $q_t b_t$ units of goods in period t and promises to pay, conditional on not defaulting, $\delta^{n-1} b_t$ units of goods in every future period $t+n$. These contracts allow long maturity debt within a tractable structure with a single state variable that incorporates all past borrowing.

Each period, the borrower has total payments due a_t, which consist of all the obligations from past borrowing and defaulted debt. It chooses which fraction of these obligations to partially default on d_t and new borrowing b_t. The bond price is a function $q(a_{t+1}, d_t, z_t)$ that depends on the total debt due the following period a_{t+1}, partial default today d_t, and the endowment z_t because these variables determine future defaults. Consumption is constrained by income less total debt due net of default, $a_t(1-d_t)$, plus new borrowing:

$$c_t = y_t - a_t(1-d_t) + q(a_{t+1}, d_t, z_t) b_t. \tag{3}$$

Default carries a direct resource cost that is increasing in the intensity of the default and depends on the shock. Income after default is $y_{t+1} = z_{t+1}\Psi(d_t, z_{t+1}) \leq z_{t+1}$, with $\Psi_d(d_t, z_{t+1}) \leq 0$.

A fraction κ of the debt defaulted on, $d_t a_t$, remains as the future debt obligations. Such fraction is annuitized into perpetuities with decay δ such that $(1-\delta)$ of $\kappa d_t a_t$ is added to next period's obligations a_{t+1}. Note that when fraction $\kappa = 1$, the defaulted debt is accumulated at no discount except for the interest rates lost from periods of positive default.

The total debt obligations a_{t+1} next period not only include the accumulation of the defaulted debt $(1-\delta)\kappa d_t a_t$ but also, as is standard with these perpetuity contracts, include the fraction δ of the obligations this period a_t and the new borrowing b_t. The evolution of the total debt due is hence given by

$$a_{t+1} = \delta a_t + (1-\delta)\kappa d_t a_t + b_t. \tag{4}$$

Debt due a_{t+1} is high when past borrowing is high $\delta a_t + b_t$ or partial default is high d_t.

3.2 Recursive Formulation

We focus on recursive Markov equilibria and represent the borrower's infinite horizon decision problem as a recursive dynamic programming problem and the lenders' values also as a recursive functional equation. The state vector of the model consists of three variables $\{a, y, z\}$: a is total debt due, y is the income of the economy, and z is the endowment shock, which we need to keep track of because of the Markovian structure of shocks.

Borrower The recursive problem of the borrower whose state is $\{a, y, z\}$ is to choose new borrowing b, partial default d, and consumption c to maximize its value

$$V(a, y, z) = \max_{b,d,c} \{u(c) + \beta \sum_{z'} \pi(z', z)V(a', y', z')\} \tag{5}$$

subject to the budget constraint that indicates that consumption is income minus the net debt repayment,

$$c = y - a(1 - d) + q(z, a', d)\ b, \tag{6}$$

the law of motion for the evolution of debt due that incorporates defaulted debt and new borrowing

$$a' = \delta a + (1 - \delta)\kappa da + b, \tag{7}$$

the evolution of the income that depends on partial default and on the shock,

$$y' = z'\Psi(d, z'), \tag{8}$$

and the constraint that default cannot exceed the debt due and is weakly positive $0 \leq d \leq 1$.

This problem determines the optimal borrowing and partial default policy functions $b(z, a, y)$, $d(z, a, y)$, the evolution of the debt due $a'(z, a, y)$, and the income state $y'(z', z, a, y)$, as well as the consumption function $c(z, a, y)$.

The bond price is a function $q(a', d, z)$ that compensates for the possible losses due to default and depends on debt due a', partial default d, and the shock z because, as shown below, these variables determine the future expected losses from default. As we will see below, default risk is higher when debt due is high and when expected income is low, which is determined by d and z. Current default d affects the bond price for new borrowing through two channels. First, the bond price function directly depends on default d because high default tends to decrease future income, which increases default risk. Second, and more importantly, current default increases debt due a' as the defaulted debt accumulates, and high debt increases default risk. This effect makes any additional borrowing b more expensive because even absent additional new loans, the debt due is high with the defaulted debt.

Lenders There are many identical, competitive risk-neutral lenders that discount time at rate $1/R$. While the aggregate state of the arrangement is $\{z, a, y\}$, the individual state of any measure zero lender is $\{x\}$, which is the debt due on its perpetuity holding. The payoff to a lender of holding $\{x\}$ is given by the value function $\Omega(z, a, y, x)$ as follows:

$$\Omega(z, a, y, x) = x\,(1 - d(z, a, y)) + \frac{1}{R}\,\sum_{z'} \pi(z', z)\Omega(z', a', y', x'), \tag{9}$$

and the evolution of its individual debt holding, which is governed by the structure of the perpetuity and by the borrower's partial default decision rule:

$$x' = \delta x + (1 - \delta)\kappa d(z, a, y)\; x.$$

Each lender takes as given the borrower's decision rules $b(z, a, y)$, $d(z, a, y)$, $a'(z, a, y)$, and $y'(z', z, a, y)$.

We can establish that $\Omega(z, a, y, x)$ is linear in x and solve it explicitly by the guess-and-verify method. Specifically, it takes on the form

$$\Omega(z, a, y, x) = x\; H(z, a, y), \tag{10}$$

where the aggregate state-dependent coefficient $H(.)$ solves
$H(z, a, y) =$

$$[1 - d(z, a, y)] + \frac{1}{R}\,[\delta + (1 - \delta)\kappa d(z, a, y)]\,\sum_{z'} \pi(z', z)H(z', a'(z, a, y), y'(z', z, a, y)). \tag{11}$$

The expression $H(z, a, y)$ is the value to a claim of one unit of debt. When a lender owns one unit of coupon a, it gets this period the portion $(1 - d(z, a, y))$ of the coupon. A claim today also has value tomorrow because debt contracts are perpetuities and because a fraction of the defaulted debt is paid in the future. Tomorrow, the lender's holdings of a pay δ plus the annuitized value of accumulated defaulted debt $(1 - \delta)\kappa d(z, a, y)$ times the next period's value. The expected value for every unit of debt tomorrow is precisely $E\{H(z', a', y')|z\}$.

Equilibrium Competition among lenders implies that every new loan to the borrower makes zero profits in expected value. This means that the bond price function is

$$q(z, a', d) = \frac{1}{R} \sum_{z'} \pi(z', z) H(z', a', y'). \tag{12}$$

The bond price is a schedule that depends on (z, a', d) because these three variables determine the probability distribution of the arguments in H, namely, (z', a', y'), which determine the loss from default. The shock z provides information for the distribution of z'. The debt due tomorrow a' affects default by the decision rule for d'. The bond price function also depends on the intensity of default today d because the income state tomorrow y' is directly affected by it.

We now define the equilibrium for this economy.

Definition 1 *A recursive Markov equilibrium consists of (i) the borrower's decision rules for new borrowing $b(z, a, y)$, partial default $d(z, a, y)$, and consumption $c(z, a, y)$, which induce rules for debt due $a'(z, a, y)$ and income $y'(z', z, a, y)$; (ii) the value for the lenders $\Omega(z, a, y, a)$, and (iii) the bond price function $q(z, a', d)$ such that*

1. *Taking as given the bond price function $q(z, a', d)$, the borrower's decision rules satisfy the borrower's optimization problem in (5).*
2. *Taking as given the borrower's decision rules, the value for the lenders satisfies (10) and individual lenders' holdings of debt equal the borrower's debt due $x = a$.*
3. *The bond price function satisfies (12).*

Using the definition for $H(z, a, y)$, the bond price function solves the functional equation

$$q(z, a', d) = \frac{1}{R} \sum_{z'} \pi(z', z) \left(1 - d' + (\delta + \kappa(1 - \delta)d')q'(z', a'', d')\right), \tag{13}$$

where the future borrower's choices and states are evaluated using its decision rules $d' = d(z', a', y')$, $a'' = a(z', a', y')$ coming from problem (5). It is useful to map the bond price function $q(z, a', d)$ to an interest rate spread function $s(z, a', d)$. Spreads equal the difference between the yield to maturity of the bond and the risk free rate. Given the long-term debt structure, the spread function is given by

$$s(z, a', d) = (1/q(z, a', d) + \delta) - (1 + r). \tag{14}$$

3.3 Characterization of Equilibrium

In this section, we study the trade-offs in the joint choice of borrowing b and partial default d by analyzing the optimality conditions for the sovereign. In these derivations, we assume that the price function $q(z, a', d)$, the value function $v(z, a, y)$, and the default cost function $\Psi(d, z')$ are differentiable. For notational convenience, we also define the bond price as a direct function of the decisions b and d by $Q(z, b, d, a) \equiv q(z, \delta a + b + (1-\delta)\kappa da, d)$.

Using compact notation, we can then write the first-order conditions of problem (5) with respect to b and d as follows:

$$b: \; u'(c)[\underbrace{q + Q_b b}_{\text{borrowing gain}}] = \beta E u'(c') \underbrace{\Lambda(d', q')}_{\text{debt burden}} \tag{15}$$

$$d: \; u'(c)[\underbrace{a + Q_d b}_{\text{partial default gain}}] = \beta E u'(c')[\underbrace{(1-\delta)\kappa a \Lambda(d', q')}_{\text{debt burden}} - \underbrace{z' \Psi_d}_{\text{default cost}}], \tag{16}$$

where the term $\Lambda(d', q')$ captures the reduction in resources for c' from a one unit increase in a' that results from increasing b and d such that

$$\Lambda(d', q') \equiv \underbrace{1 - d'}_{\text{current repayment}} + \underbrace{(\delta + (1-\delta)\kappa d') q'}_{\text{future repayment}}.$$

These optimality equations illustrate how the borrower can transfer future resources to the present by borrowing with new loans b or by defaulting d. They equate the marginal gain in utility from borrowing or partial default to the marginal reduction in utility from repaying the future debt burden and experiencing default costs. Effectively, b are d are different ways of altering the debt position, and each has its costs and benefits. Partial default acts like expensive debt in this model.

The left-hand-side (LHS) terms in square brackets in equations (15) and (16) capture the gains from borrowing and defaulting, which depend on the bond price function. The right-hand-side (RHS) terms of both expressions illustrate the costs of repaying the current and future coupon payments as well as the accumulated defaulted debt, which are evaluated at the price q'. The costs also incorporate the resource costs from default.

Consider first the optimality condition for b from equation (15), which is similar to that arising in many dynamic sovereign default models. The LHS contains the marginal benefit from borrowing one unit of b, which increases consumption by q but is discounted by the decrease in the price with more borrowing, $Q_b = q_{a'} < 0$. In our model, as is typical in models of sovereign default, the resources raised by borrowing are capped by a Laffer curve; at the top of the Laffer curve the marginal benefit of borrowing is zero. Such Laffer curve arises because default risk limits the possibility of inter-temporally transferring resources with loans.

The RHS in equation (15) contains the discounted marginal cost from borrowing, which contains the terms in $\Lambda(d', q')$ corresponding to repaying the next period's coupon as well as refinancing the future coupons and the defaulted debt in arrears. The coupon payments a' are discounted by d' as only this fraction will be repaid. This discount, however, is not full as $\kappa d'$ is accumulated over time and evaluated at price q'. The accumulation of the defaulted debt allows the borrower to create long-term debt regardless of the duration of the debt, including when $\delta = 0$ and the stock of debt is short term. An implication of our set up is that dilution and commitment problems inherent in environments with long-term debt, also arise when defaulted short-term debt accumulates over time.[5]

Consider now the optimality conditions for partial default d from equation (16). This condition contains the trade-off for partially defaulting on the debt, accumulating the defaulted debt, and experiencing the default costs. The LHS of this equation says that defaulting a d fraction of a is beneficial as consumption is increased by a, but this benefit is discounted by the fact that the price for b falls. The price $q(a', d, z)$ falls as d increases because of the direct effect $q_d < 0$ and also because d increases a' and $q_{a'} < 0$. The term $Q_d = (1-\delta)\kappa a q_{a'} + q_d$ captures these price effects. Raising resources through partial default does not have the acquiescence of lenders and these resources are not capped by a Laffer curve but by the total level of debt due a. The costs of partially defaulting are on the RHS. One component arises because the debt defaulted on is annuitized and accumulates with the discount of κ. The resulting debt burden from partial default after annuitization and discounting $(1-\delta)\kappa a$ is the same as for borrowing and given by $\Lambda(d', q')$. Partial default also carries an additional cost in terms of reduced resources. Such default cost is encoded in $\Psi_d < 0$. Note that partial default tends to be more advantageous for larger levels of debt a as the direct benefit of defaulting is increasing in a while the direct cost of defaulting, $-\Psi_d$, is independent from a.

By combining (15) and (16), we derive the following condition, which equates the expected returns

[5] Aguiar and Amador (2018) and Aguiar and Amador (2019) establish that the Eaton and Gersovitz (1981) model with short-term debt features uniqueness of equilibrium and can be represented as an optimal contracting problem while with long-term debt the environment features multiplicity with inefficient equilibria. Such contrasting results with short versus long-term debt rely on default eliminating all debt.

of borrowing and partial default weighted by marginal utility as follows

$$Eu'(c')\underbrace{\frac{\Lambda(d',q')}{q+Q_b b}}_{\mathsf{R}^b} = Eu'(c')\underbrace{\frac{(1-\delta)\kappa}{1+Q_d b/a}\left[\Lambda(d',q')+\frac{z'}{(1-\delta)\kappa a}(-\Psi_d)\right]}_{\mathsf{R}^d}, \tag{17}$$

where R^b and R^d are the returns to be paid out from borrowing and partially defaulting, respectively.

The implication from equations (17) and (15) is that the weighted expected returns from borrowing and partial default are equated and equal to current marginal utility $u'(c) = \mathsf{R}^b = \mathsf{R}^d$ when these two choices are positive and interior. These conditions also imply that when partial default is zero $d = 0$, it must be that $\mathsf{R}^b \leq \mathsf{R}^d$ such that the return to be paid from borrowing is lower than that from partially defaulting.

The shape of the price function and the default costs are the determinants of the optimal portfolio of b and d. A positive b is an attractive choice when the price q is high and not too steep, such that $|q_{a'}|$ is small, which means low and insensitive default probabilities. Here b is attractive because the marginal increase in consumption with b is high, and any future default costs are minimized. A positive d becomes attractive only when q is low and steep. Here paying the default costs encoded in $\Psi_d < 0$ can be optimal for increasing consumption today at a rate of 1. The extra costs from d in terms of the price of new loans can be eliminated by setting b close to zero. The attractiveness of d also depends on the rate at which this default is accumulated κ. Partial default is most attractive when debt relief is high and little of the defaulted debt is accumulated for future repayment, which happens when κ is low.

Condition (17) also highlights that the shape of the penalty function Ψ and in particular the derivative $(-\Psi_d)$ are important for when default will be partial, total, or zero. As we will see in the quantitative section, when $(-\Psi_d)$ is increasing with d, default is most likely partial because here it becomes increasingly costly, so it is optimal to stop defaulting before $d = 1$.

3.4 Default Episodes: Length, Haircut, Maturity Extensions

We define a *default episode* as the period of time with consecutive positive partial default, which is the same definition we use in the data. The default episode *length* is the number of periods with positive partial default. The long-term structure of the bonds in our model and the flexible default choices provide a meaningful setting to measure haircuts and maturity extensions from default episodes.

We measure the duration of debt with the standard Macaulay duration. Debt duration is the weighted average of the time of each coupon payment, with weights equal to the fraction of the bond's value on each payment. The duration of the perpetuity bonds with decay δ is

$$\text{dur}_\delta = \frac{1}{q}\sum_{n=1}\left(n\frac{\delta^{n-1}}{(1+r)^n}\right) = \frac{1}{q}\frac{1+r}{(1+r-\delta)^2} = \frac{1+r}{1+r-\delta}.$$

In this calculation, we discount streams with the risk free rate r and use the expression for the discount price $q = \frac{1}{1+r-\delta}$.

We follow the methodology in Benjamin and Wright (2013) and Cruces and Trebesch (2013) to compute haircuts and maturity extensions of default episodes. This methodology compares the present values of streams of old defaulted debt instruments to new restructured debt instruments. We discount the streams at the risk-free rate. To develop the expressions for haircuts and maturity extensions, consider a default episode that starts in period 1 and ends in period N. The face values of the defaulted debt during the episode are given by $\{d_0a_0, d_1a_1, d_2a_2, ...d_Na_N\}$, where default is zero in period $N+1$ by construction, $d_{N+1} = 0$. The present value and the duration of the old defaulted debt instruments DD from time zero are

$$\text{value}(DD) = \sum_{t=1}^{N}\frac{d_ta_t}{(1+r)^{t-1}}, \qquad \text{dur}(DD) = \frac{1}{\text{value}(DD)}\sum_{t=1}^{N} t\frac{d_ta_t}{(1+r)^{t-1}}.$$

The new restructured debt instruments in exchange for the old defaulted debt instruments correspond to the arrears paid through the default episode and the terminal value of these arrears. The face values of the debt due $\{a_2, a_3, ..., a_{N+1}\}$ contain both the coupons from legacy debt that have not been defaulted on as well as the arrears that accumulate as new debt instruments $\kappa d_ta_t(1-\delta)$. This latter arrears component can be expressed recursively as $a_t^{ND} = (1-\delta)\kappa d_{t-1}a_{t-1} + \delta a_{t-1}^{ND}$ for $t = 2, ..., N+1$ with $a_{N+1}^{ND} = 0$. The new debt instruments in each period correspond to the arrears paid during the default episode net of partial default, $(1-d_t)a_t^{ND}$, plus the value of the terminal coupon a_{N+1}^{ND}. Their present value is given by

$$\text{value}(ND) = \sum_{t=2}^{N}\frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N}\frac{1+r}{1+r-\delta}.$$

The duration of the new debt instruments is therefore

$$\text{dur}(ND) = \frac{1}{\text{value}(ND)} \sum_{t=2}^{N} t \frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N} \left(N \frac{1+r}{1+r-\delta} + \frac{1}{(1-\frac{\delta}{1+r})^2} \right)$$

We can now define the haircut and the maturity extension of the default episode. We define the *haircut* as the ratio of the arrears paid to the defaulted debt and the *maturity extension* as the difference between the duration of the arrears paid and the defaulted debt:

$$\text{haircut} = 1 - \frac{\text{value}(ND)}{\text{value}(DD)} \tag{18}$$

$$\text{maturity extension} = \text{dur}(ND) - \text{dur}(DD). \tag{19}$$

The debt haircut and maturity extensions depend in our model on the path and intensity of partial default during the episode, the length of the episode, as well as on the model parameters and in particular the discount parameter controlling debt relief from default, κ. To develop some intuition for these forces, we present the recovery rate $= \frac{\text{value}(ND)}{\text{value}(DD)}$, which is the complement of the haircut, for a two-period episode $N = 2$. The recovery rate in this case is

$$\text{recov}_{N=2} = \frac{(1-\delta)\kappa}{1+r-\delta} \left[1 - \frac{\left(1 - \frac{\delta}{1+r}\right) d_2 d_1 a_1}{d_1 a_1 + \frac{d_2 a_2}{1+r}} \right].$$

The first component of the recovery rate $\frac{(1-\delta)\kappa}{1+r-\delta}$ is exogenous and depends on the discount κ. The recovery rate has a second multiplicative component that is endogenous as it depends on debt, partial default, and length during the episode. When $d_2 = 0$, the default episode is effectively a one-period episode with an exogenous haircut that only depends on κ. In a two-period episode, however, both partial default rates d_1 and d_2 reduce recovery. The recovery tends to decrease with the length of the episode as default on older arrears is compounded over more periods of time. Even if default is total, that is, $d_1 = d_2 = 1$, there is positive recovery since the episode ends with repayments of the accumulated defaulted debt.

4 Quantitative Results

We now evaluate the quantitative properties of our model and compare them to the data on sovereign defaults in emerging markets. We present the parameterization of the model and illustrate its mechanics by analyzing the resulting decision rules and impulse response functions. We compare the implications of our model for partial default and default episodes to the data and perform a counterfactual analysis.

4.1 Specification and Parameterization

Functional Forms. We first specify the functional forms used in the quantitative analysis. The utility function of consumption is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with σ being the constant coefficient of risk aversion. We assume a discrete process for shocks to output that approximate the autoregressive process

$$\log z' = \rho \log z + \eta',$$

where the innovations $\eta' \sim N(0, \sigma_\eta)$. We approximate z with a 10-state Markov chain following the discretization methods in Tauchen (1986).

Output in our model depends on the shock and also on default costs in such a way that output next period $y' = z'\Psi(z', d)$. The function Ψ describes the output costs from defaulting which depend on the shock z' as well as on the default intensity d. Following the quantitative sovereign default literature (e.g., Arellano (2008) and Chatterjee and Eyigungor (2012)), the output costs of default are assumed to be increasing and convex in the shock z'. Specifically, we assume that default costs are only realized for sufficiently high z above a threshold z^* and are then linearly increasing in z with a slope ϕ_1. We also assume that the costs are increasing and convex in partial default d with slope parameter ϕ_0 and curvature parameter γ. The functional form we consider is

$$\Psi(z', d) = (1 - \phi_0 d^\gamma)(1 - \hat{\phi}_1(z' - z^*)), \tag{20}$$

where $\hat{\phi}_1 = \phi_1$ if $d > 0$ and $z' > z^*$ and zero otherwise.

We now describe the parameter values. We choose parameters so the variables in the model are measured at a quarterly frequency. We use existing studies to assign some parameters and use a moment-matching exercise for the remaining ones.

Assigned Parameters. The parameters assigned directly are risk aversion, the risk-free interest rate, and the decay parameter controlling debt duration. We set risk aversion σ to 2 and the risk-free rate r to 1% quarterly. These are common values in business cycle studies and also in

Arellano (2008). The decay parameter for the perpetuity contracts δ is set so that the duration of the debt in our model equals 5 years, which is similar to the average duration of debt in emerging markets. The bottom section of Table 4 displays these choices.

Moment-Matching Exercise. The rest of the parameters are set in a moment-matching exercise. We choose the eight parameters

$$\Theta = \{\phi_0, \phi_1, \gamma, z^*, \beta, \kappa, \rho, \sigma_\eta\}$$

to best match ten moments from emerging markets data. The first five moments summarize the empirical distribution of partial default. These targets are the frequency, overall mean, and standard deviation of partial default, and the mean of small defaults as reported in Tables 1 and 3. The next four moments are the mean debt-to-output ratio, the standard deviations of debt to output and of sovereign spreads, and the correlation of spreads with output.[6] Finally, we include as targets the standard deviation and persistence of output in Argentina as reported in Arellano (2008).

To compare the model moments to the data targets, we simulate the model and aggregate the quarterly model time series to an annual frequency. We simulate the model for 750,000 periods and discard the initial 10% of observations. This long simulation approximates the limiting distribution across the states $\{z, a, y\}$. We construct annual series from quarterly simulations as follows. The annualized series for debt due a_t, total debt $q_t a_t$, defaulted debt $d_t a_t$, and output y_t are the sum of the series from the previous four quarters such that $x_t^{\text{an}} = \sum_{j=0}^{3} x_{t-3+j}$ for $x_t = \{a_t, d_t a_t, y_t\}$. Annualized partial default is $(da)_t^{\text{an}}/a_t^{\text{an}}$, and total debt to output is evaluated with risk-free rates. We construct the annualized spreads from the bond price following equation (14) such that $s_t = (1/q_t + \delta)^4 - (1+r)^4$.

We seek parameter values that minimize the sum of squared proportional deviations of the model moments from their corresponding data moments. In the appendix, we provide the detailed algorithm we use to compute the model and the details of our moment-matching exercise. The top section of Table 4 summarizes the target moments and the parameter values.

The threshold parameter z^* is such that default costs are positive when the shock is above 93%

[6] These eight moments are computed as means across the 38 emerging countries in Section 2 of the country-specific variable. For example, the target correlation of output and spreads is the mean correlation across these countries.

Table 4: Parameter Values

Parameters from Moment Matching		
Default costs	$\phi_0 = 0.04$, $\phi_1 = 0.206$	Partial default:
	$\gamma = 1.621$	frequency, mean, st. dev., small
	$z^* = 0.933 \times \overline{z}$	Debt to output: mean, st. dev.
Discount factor	$\beta = 0.987$	Spread: st. dev.
Recovery fraction	$\kappa = 0.926$	Corr. spread with output
Shock process	$\rho = 0.928$ and $\sigma_\eta = 0.028$	Output st. dev., persistence
Assigned Parameters		
Risk aversion	$\sigma = 2$	Standard business cycle models
Risk-free rate	$r = 1\%$	Quarterly risk-free rate
Decay parameter	$\delta = 0.96$	Debt duration 5 years

of the mean. The value of ϕ_0 says that the loss of output due to the intensity of default can be 4% at most when default is full. The value of γ greater than unity implies that default costs are convex in the default intensity. The recovery fraction κ implies a 7% discount in the face value of coupon payments.

The eight parameters of the moment-matching exercise have consequences for all of the ten targeted moments. However, some of the parameters appear to be more closely related to certain moments than others. The parameters controlling the z shock are primarily related to the persistence and volatility of output. The persistence and volatility for productivity have to be slightly smaller than those for the targeted output process. The discount rate β is closely connected to the debt-to-output ratio. The two parameters describing the fixed component of the penalty for default, ϕ_1 and z^*, have an impact on the frequency of defaults and the size of defaults at the low quantile. A larger z^*, which waives the fixed cost at low shock levels, raises the frequency of default and reduces the size of partial defaults. A larger ϕ_1 works in the opposite direction; a bigger cost discourages very small partial defaults. The parameters γ and ϕ_0, describing the penalty for default that depends on its intensity, mainly affect the size of partial default and its volatility, as well as the correlation of spreads with output. Large γ decreases the size and volatility of partial default and increases the correlation of spreads with output. Large ϕ_0, in contrast, increases the size and volatility of partial default yet also increases the correlation of spreads with output. Finally, the parameter that controls the accumulation of defaulted debt κ increases the volatilities of spreads and debt and decreases the correlation of spreads with output.

4.2 Moments in Model and Data

Table 5 presents the results from the moment-matching exercise. The model reproduces many of the properties of partial default. In the model and the data, partial default is positive about one-third of the time. On average, default occurs on one-third of the debt repayment due; although in the model, the mean of partial default is a bit lower, the outcome is reasonably close.[7] Partial default, conditional on being positive, is volatile and features many small defaults in the model and the data. In the model, the standard deviation of partial default is 20%, and the average default of small defaults is 7%. These moments are only moderately higher than the data counterparts of 16% and 3%.

Debt is high and volatile in both the model and the data. In the model, the mean of debt to output is 36% with a standard deviation of 18%. These moments are close to the data counterparts of 37% and 21%. In terms of the interest rate spreads, the model produces a volatility similar to the data and a spread that is negatively correlated with output, with a correlation comparable to the empirical counterpart. The model also matches well the resulting series for output in terms of its persistence and standard deviation.

Table 5 also reports additional untargeted moments from our model and compares them with the data. The model generates a mean level of debt due relative to output of 7%, which is equal to the average in the data.[8] As shown in Chatterjee and Eyigungor (2012), models with long-term debt, as our model, are able to reproduce both total debt levels and debt service levels. The model generates a positive correlation between debt and spreads, which is also a feature of the data. High debt increases default probabilities in both the model and the data.

In terms of mean spreads, the model predicts a mean spread of 1.2%, which is lower than the average in the data of 5.3%. The discrepancy arises partly because we are matching default frequencies and partial default, which determine the expected losses for creditors and are modeling creditors as risk neutral. The wedge between spreads and expected default losses is related to the credit spread puzzle in corporate bonds where default losses cannot explain the observed spreads, as explored in Chen, Collin-Dufresne, and Goldstein (2009). Longstaff et al. (2011) find that a similar wedge arises for sovereign spreads. They find, for example, that about 64% of sovereign

[7] On the raw quarterly series, the frequency of default and the mean partial default are 26% and 40%, respectively. Therefore, that default is partial clearly does not come from time aggregation over periods with full default; time aggregation reduces it only slightly. Similarly, the large frequency of default does not appear to be driven by time aggregation either, to any large extent.

[8] In the data, we measure debt service due as the sum of debt service and arrears.

Table 5: Moments in Model and Data (%)

	Data	Model
Target Moments		
Partial default frequency	35	34
Partial default mean	35	31
Partial default st. dev.	16	20
Small partial defaults	3	7
Debt-to-output mean	37	36
Debt-to-output st. dev.	21	18
Spread st. dev.	3.6	3.9
Corr. (output, spread)	-30	-32
Output persistence	0.93	0.93
Output st. dev.	0.08	0.08
Other Moments		
Debt service to output	7	7
Corr. (debt, spread)	24	47
Spread mean	5.3	1.2

spreads can be attributed to a single global factor that contains large risk premia. The spread more directly comparable to our model's corresponds to the remaining proportion, or 1.9% (5.3% $\times$0.36), which is much closer to the model's prediction.

4.3 Decision Rules and Impulse Responses

We study the model's decision rules and impulse responses for partial default, interest rate spreads, borrowing, and consumption. Analyzing our theory is useful for interpreting the quantitative implications of the model for the comovements of partial default with spreads, debt and output, and the properties of default episodes in relation to the data.

4.3.1 Decision Rules

The left panel in Figure 3 plots the default decision $d(z, a, y)$ as a function of the level of debt due a for two levels of the shock $z = \{z_5, z_3\}$, a median and a lower level. We scale the total debt due by the risk-free discount price $1/(1 + r - \delta)$ relative to mean output. The level for current output in the decision rules equals the level of the shock $y = z$. The figure shows that partial default increases with the level of debt due and decreases with the level of output.

The figure shows that when debt is small enough, partial default is zero; as debt increases, partial default increases; and when debt is large enough, default is total.[9] Comparing across the two curves also demonstrates that partial default decreases monotonically with output, given debt due.

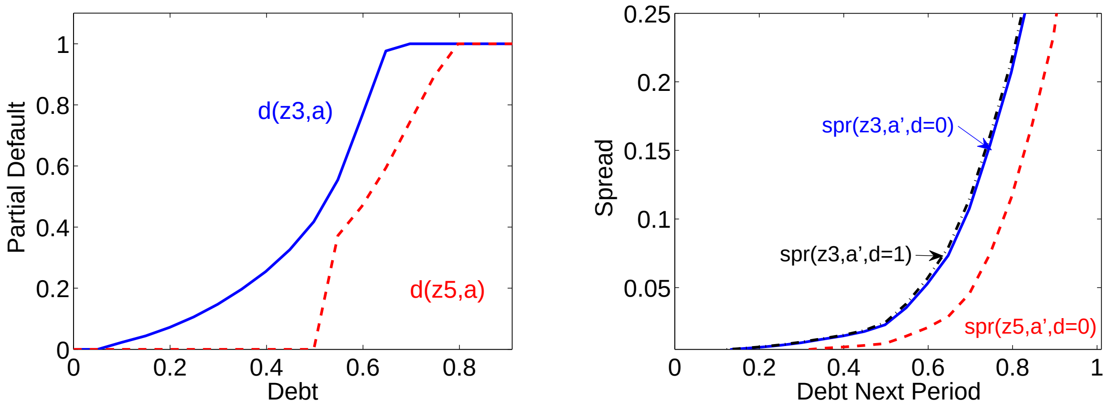
Our model then generalizes the results from models of full default that feature default being more likely with high debt and low output (Arellano (2008)) to the case of partial default. In our model, not only is positive partial default more likely with high debt and low output, but also the intensity of default increases with high debt and low output.

Consider now the spread schedule $s(z, a', d)$ in the right panel of Figure 3. We plot the spread schedule as a function of the next period's debt due a (also scaled by the risk-free discount price and relative to mean output) for the same two levels of the shocks z as above, with partial default equal to zero $d = 0$. We also plot an additional curve in the low output state for positive default, $d = 1$, in order to isolate the impact of defaulting on spreads coming solely from the future output loss. With long-term debt and accumulation of arrears, the future debt due depends not only on borrowing b but also on today's debt due a and default d, so that $a' = \delta a + b + (1 - \delta)\kappa da$. Hence, the spread schedules in the figure with zero and positive partial default have different underlying legacy debt δa and borrowing b because in the figure we condition on the same debt due tomorrow a' across schedules.

The figure shows that spreads increase as future debt due increases. Low levels of future debt due have zero spreads because predicted partial default is low. The increase in spreads as a function of future debt due is fairly smooth in our model because, as discussed above, default intensity varies gradually. The figure also shows that spreads are higher when output is low today, as this indicates a higher likelihood of low output in the future and hence of higher partial default. Finally, the spread schedule also depends on the partial default decision today d, the reason being that higher d induces a larger output loss next period, and hence the possibility of higher partial default in the future. The direct output loss from partial default in the future, however, appears to have only a modest impact on spreads.

We now turn to the implications of the functions described for the observations of interest.

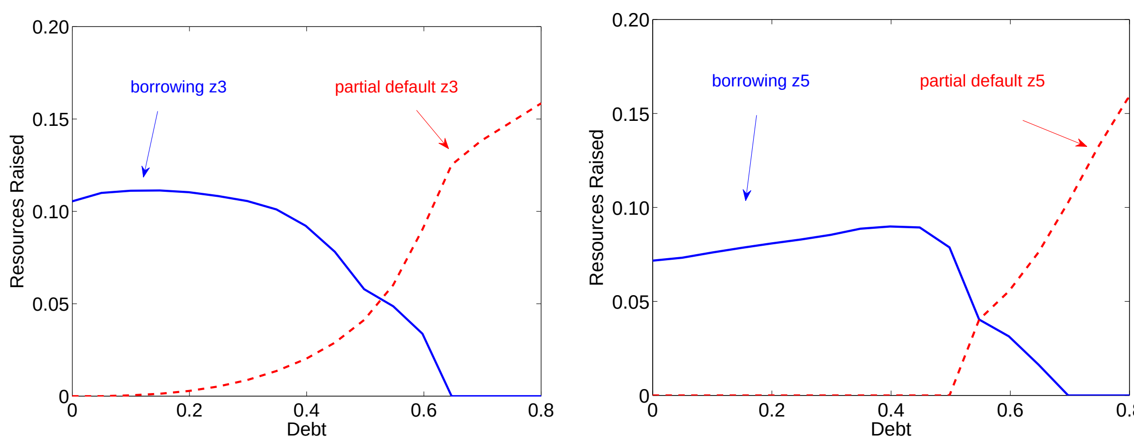
[9] Note that for the low shock, the change in default is continuous since it does not carry a fixed cost. For the larger shock level, default presents a discontinuity where partial default jumps to a positive value, reflecting the fixed cost from the default in this case.



(a) Partial Default Decision Rule (b) Spread Schedule

Figure 3: Partial Default and Spreads

Regarding correlations, the default policy rule suggests a direct impact of debt on default intensity, which tends to impart a positive sign to the correlation between these two variables. From the spread function, since the influence of default via the penalty is weak, the correlation between default intensity and spread must reflect the two common factors, output shocks and debt, driving a positive comovement between the two variables. These policy rules, however, do not account for the feedback effect of default on the accumulation of debt and hence spreads, which, as we describe below, is important for the time series comovements.



(a) Low Shock (b) High Shock

Figure 4: Portfolio of Borrowing and Default

Next we consider the policy rules for borrowing. As discussed above, in our model the country can transfer future resources to the present by borrowing qb or by partially defaulting da. Such a portfolio view for borrowing and default is evident from the Euler equations (15) and (16). The optimal portfolio mix changes with debt and income. Figure 4 plots the total resources raised with borrowing $q(z, a', d,)b(z, a, y)$ and partial default $d(z, a, y)a$ (relative to mean output) as a function of debt due a for the shocks $z = z_3$ and $z = z_5$ with $y = z$. When debt is low, the country uses only borrowing to raise external resources. As debt rises, the portfolio shifts toward partial default, with borrowing declining sharply and partial default rising toward 100%. The reason behind the portfolio switch is the shape of the spread schedules encoded in $q(z, a', d)$. As debt due a increases, the interest rate spread for additional borrowing b increases because default risk is higher with higher total debt due tomorrow $a' = \delta a + b$. The states with high a are associated with severely restricted credit access, which makes it optimal for the country to use default as a way to raise resources. The policy rules for the optimal portfolio mix suggest that during default episodes, borrowing and default will move in opposite directions as long as debt accumulates within those episodes.

Figure 4 also shows that high z shocks are associated with more borrowing when initial debt is less than about 50% of output. For higher levels of debt, however, this pattern reverses and borrowing is higher for high levels of the shock. Such procyclical borrowing is typical in sovereign default models because of the more lenient price schedules in booms.

4.3.2 Impulse Responses

We now analyze the equilibrium time series of partial default and spreads, as well as borrowing, total debt due, and output, in response to shocks. We construct impulse response functions in our nonlinear model as follows. We simulate $1,000,000$ paths for the model for $1,000$ periods. From periods 1 to 500, the aggregate shocks follow their underlying Markov chains. In period 501, we reduce the value of productivity z in each simulation. From period 501 on, the aggregate shocks follow the conditional Markov chains. The impulse responses plot the average, across the $1,000,000$ paths, of the variables from period 500 to 600. The time series in this impulse responses are at a quarterly frequency.

Figure 5 contains impulse responses for a small and large shock in period 0. For the small shock, we reduce the values of z by one grid index, and for the large shock, we reduce them by two grid indexes. The top left panel of the figure plots the paths for the shocks z in percentage deviations. The small shock is a decline of 4.3% or about half of the standard deviation, and the large shock is a decline of 8.6%. The paths for output essentially mirror the paths for productivity as the

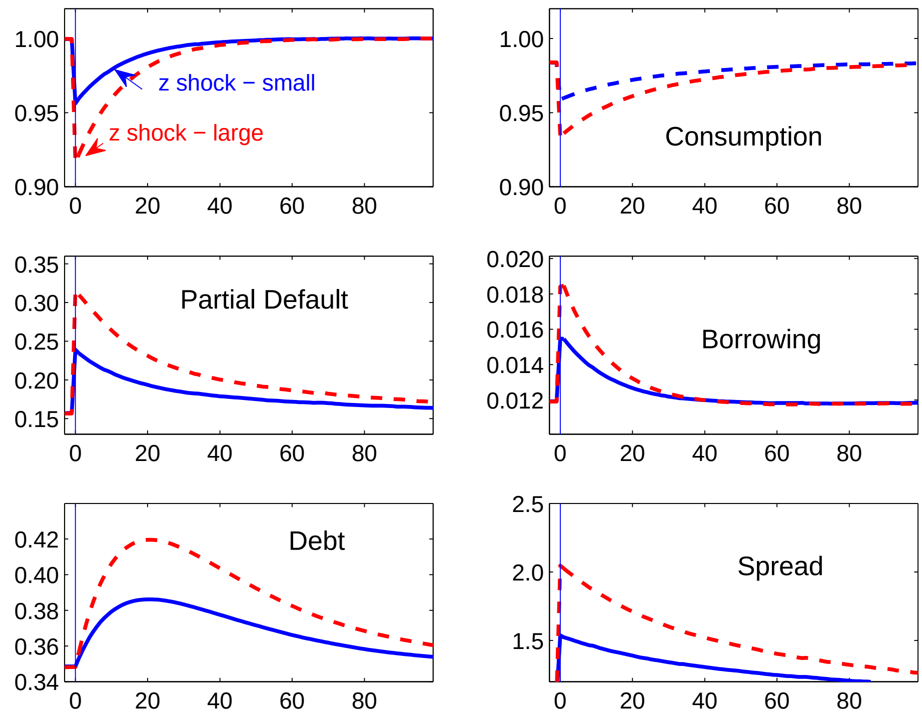


Figure 5: Impulse Response Functions

impact of default costs is not large.[10]

We first describe the responses of variables to the small shock, which are plotted with solid lines in Figure 5. The right top panel shows the path for consumption, logged and in percentage deviations. Consumption on impact falls somewhat less than output but stays depressed for a longer period of time. Default risk restricts the ability of the economy to smooth consumption, which leads to a large decline on impact. Consumption remains depressed because debt accumulates and remains persistently high.

The middle panels of Figure 5 plot the paths for the portfolio choices of partial default d and borrowing b. We report borrowing scaled by $1/(1+r-\delta)$ relative to mean output. Partial default and borrowing both increase on impact in response and return to their means. Partial default increases by about 8% on impact and features a response that is more persistent than the shock. By period 40, the shock has largely recovered, while partial default continues to be elevated above trend. Borrowing also increases on impact, but the effect is much less persistent than the effect

[10]Output depends not only on the shock but also on partial default $y' = z'\Psi(z', d)$, but in these impulse response functions, output is extremely close to the shock. The main deviation is in period 1: output falls by 4.4% and 8.7% whereas the shock falls by 4.3% and 8.6%.

on partial default.

The bottom left panel plots the response for debt due a that results from the choices of d and b. We report the debt due scaled by $1/(1+r-\delta)$ relative to mean output. The stock of debt features a hump shape, increasing for the first 20 periods after the shock and then slowly decreasing. The rise in partial default and borrowing both increase total debt due; partial default delays the repayment of debt through the accumulation of arrears, and borrowing directly increases the total debt due. The impulse response of total debt is slow and quite persistent. Debt remains elevated for much longer after the shock has returned to its mean.

Finally, the bottom right panel in the figure plots the path for the interest rate spread. Spreads rise on impact about 60 basis points and decrease slowly. Spreads rise because, as discussed in Section 4.3, partial default increases with low shocks and higher debt due. The effects on spreads are also very persistent. In period 40, for example, spreads continue to be 25 basis points higher than before the shock.

Consider now the impulse response functions to the larger negative shock, shown in Figure 5 with dashed lines. The responses of the variables to the large shock have a larger magnitude and are more persistent than the responses to the smaller shock. To see this higher persistence, consider the paths for total debt. By period 60, the small and larger shocks have both recovered, yet debt is about 2 percentage points higher after the larger negative shock. The more persistent debt dynamics introduce more persistence in all the remaining variables.[11]

These impulse responses illustrate the propagation and amplification mechanisms in our model. Low shocks tighten spread schedules, making it more costly to roll off the debt. Partial default increases to alleviate the consumption decline arising from low output and tight financial conditions. New borrowing expands moderately despite spreads being high also to support consumption. The rise in partial default and borrowing both increase the future debt due and create a dynamic amplification for shocks. As debt stays elevated, partial default and spreads remain persistently high even as the shock dissipates. Moreover, the larger the shock, the stronger the propagation and amplification. Recessions in our model have long-lasting effects on the functioning of international financial markets as debt levels rise with the accumulation of defaulted debt.

Having described the impulse response functions, we now turn to discuss the implications for

[11] The impulse response functions to positive shocks are almost exactly the mirror image of the impulse response functions to negative shocks. Only the dynamics of spreads are asymmetric. Booms feature smaller reductions in spreads.

comovement for the variables of interest. As for the comovements of the intensity of partial default, the amplification effect of default on debt accumulation strengthens the correlation with debt. The comovement of default with both output and spread is very tight and should be reflected in large correlations with the anticipated sign. Regarding default episodes, the amplification in debt accumulation via default may very well imply that episodes of default end with levels of debt that are larger than they were at the start, and that episodes characterized by higher default will tend to last longer. That borrowing returns to trend faster than default suggests that during default episodes, with sustained default and increased debt, credit may become increasingly restricted.

4.4 Default Intensity

We now compare the quantitative implications of our model against the data with regard to default intensity. We show that the model can replicate the empirical comovement of partial default with spreads, debt, and output documented for the data of emerging markets in Section 2. We construct these statistics from the long simulation with annualized series.

We first analyze the mean spread, debt to output, and output across bins based on the intensity of default as measured by partial default. We partition the limiting distribution based on partial default into four bins. The no default bin corresponds to the observations with zero partial default. The small default bin corresponds to the periods with values for partial default less than its 25th percentile conditional on being positive. The medium default bin corresponds to the periods with values for partial default between the 25th and 75th percentiles. The large default bin corresponds to periods with values for partial default above the 75th percentile. In Table 6 we report the average of partial default, spreads, debt to output, and output across these default bins in both the data and the model.

The distributions of partial default in the model and data have a wide range, with zero, small, medium, and large partial defaults, although the distribution is a bit less wide in the model relative to the data. As Table 6 reports, the model and data have many observations that feature positive but small defaults as reflected by the mean of partial default for small defaults of 7% and 3% for the model and the data respectively. This moment was a target in the moment-matching exercise, already reported in Table 5. The model and data also have many observations with large defaults, as reflected by the mean of partial defaults for large defaults of 64% and 82% in the model and data, respectively. The model produces medium defaults of 27%, the same size as in the data.

In the table we report the difference in the mean spread across the four groups relative to the no default group. As explained above, our model produces a lower mean spread; the comovement of spreads with partial default, however, mirrors the data. In both the model and the data, spreads are narrower during smaller defaults. Small and medium defaults have somewhat narrow spreads in both the model and the data. The spread difference for these groups is 1% and 3% in the model and data, respectively. Large defaults have larger spreads, with a spread difference equal to 7% and 11% in the model and data, respectively.

The table also shows that debt to output monotonically increases with partial default intensity in the model, ranging from 40% to 68%, similar to the data, which ranges from 37% to 63%. Output in the model also decreases monotonically with partial default in both the model and the data, although this relation is stronger in the model.

Table 6: Partial Default, Spreads, Debt, and Output

Means (%)	No default	Partial default > 0		
		Small	Medium	Large
DATA				
Partial default	0	3	27	82
Spread difference	0	3	3	11
Debt to output	26	37	49	63
Output	1	0	-3	-6
MODEL				
Partial default	0	7	27	64
Spread difference	0	1	1	7
Debt to output	28	40	49	68
Output	4	-4	-8	-12

To analyze these comovements, we also present correlations of spreads, debt, and output with partial default, conditional on positive partial default, in the model and the data. Table 7 shows that in our model and the data, periods of large partial default are associated with high spreads, high debt, and low output. Correlations in the model have the same sign as in the data, although they are somewhat stronger.

The co-movements of the spreads, debt to output, and output with partial default reflect the response of these variables to shocks and the dynamics of debt illustrated with the decision rules and the impulse response functions analyzed in Section 4.3.

Table 7: Comovement with Partial Default

	Data	Model
Correlation with partial default		
Spread	25	59
Debt to output	44	72
Output	-12	-70

4.5 Default Episodes

As we have documented, countries go in and out of partial default recurrently. Our model has implications not only for the conditions that lead countries to partially default but also for the properties of default episodes. Recall that we define the default episode in the model the same way as we did for the data, that is, as the periods with continuous positive partial default, $d > 0$, that are preceded and followed by at least one period with zero default, $d = 0$. In this section, we explore the model's quantitative implications for length, haircuts, maturity extensions, and dynamics of default episodes.

In the model, the length of the default episode depends on the length of the recession and the degree of debt accumulation during the recession. The economy enters into partial default when hit by a large enough low shock given its level of debt. As long as the economy continues in the recession, the economy accumulates debt and remains in the default episode. When output recovers enough, the economy exits partial default and the default episode ends. In this framework, longer default episodes arise from longer recessions with large accumulation of debt.

To illustrate these mechanics, we plot the average paths for the variables of interest during default episodes. We consider paths for short default episodes lasting 3 years and long default episodes lasting 10 years. In particular, we simulate the economy for 1,000,000 quarterly periods and sort default episodes based on their length. We then draw the means of the shock, output, debt due, borrowing, partial default, and spread for short and long default episodes. The resulting paths are shown in Figure 6.

The short and long default episodes start in period 1 of Figure 6 when output is about 6% below its mean. During the short default episode, output falls one more period and then recovers. During the long default episode, output remains depressed until period 8 and then recovers. Longer episodes also feature larger declines in output than shorter default episodes. The episodes end when output recovers sufficiently. The level of output that leads to an end of the episode, however, is higher in the long than in the short episode.

Debt accumulates during the episode mainly because partial default increases and the defaulted debt accumulates. In fact, new borrowing in markets falls after a surge in the initial periods. Spreads also increase during the episode. In longer episodes, debt and spreads increase by more. The long episode requires higher output upon exit precisely because the long episode leads to larger accumulated debt.

The model dynamics of output and debt during default episodes are consistent with the empirical findings of Benjamin and Wright (2013) that study default episodes in emerging markets. They find that default episodes are more likely to start when output is below trend and that they end when output has returned to trend. They also find that debt upon exiting the default episode is no smaller than it was prior to the episode, and in many cases it is strictly larger. The dynamics of output and debt that our model generates during default episodes are consistent with these findings.

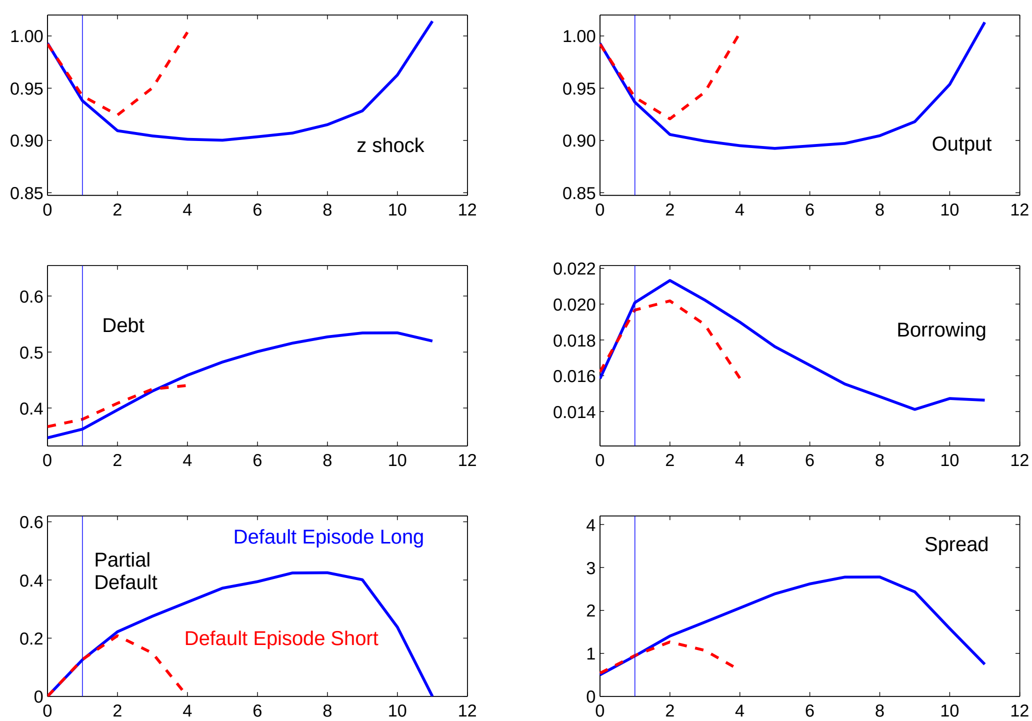


Figure 6: Default Episodes

We now compare the properties of default episode length in the model to our cross-country data.

Table 8 shows that on average default episodes last several years, in both the model and the data, although in the model they are shorter and equal to 5 years compared to 9 years in the data. The model generates a large fraction of short default episodes. About 44% of defaults in the model last less than 2 years, close to the data fraction of 38%. The model and data feature large heterogeneity in default episode length, with a very similar coefficient of variation of around 1.

In interpreting these results, it is important to note that our model abstracts from coordination failures among lenders, which is the leading theory for the length of default episodes, as in Benjamin and Wright (2013). These authors show that coordination failures can account quantitatively for large delays in resolving default episodes. Their framework, however, is unable to generate the skewness of the length distribution that contains a large fraction of short default episodes. We view our theory of default episode length as complementary to the one based on coordination failures.

In terms of debt dynamics, we compare the evolution of debt to output during default episodes to the one in the data. Our model features the hump-shaped evolution of debt during default episodes present in the data. In the model, debt to output in the period before the default episode is equal to 37%. At the start of the default episode, debt rises to 40% and continues to grow. In the middle of the episode, debt is 45%, and toward the end of the episode, it falls modestly to 44%. In our model, as in the data, default episodes do not lead to reductions in debt, and debt ratios continue to rise during the default episode.

Table 8: Default Episodes

	Data	Model
Mean episode length	9	5
Fraction of short episodes (≤ 2)	38	45
Coefficient of variation for episode length	1.1	1.2
Haircut (%)	36	30
Duration extension	6	5
Debt in episode (%)		
Before	35	37
Beginning	40	40
Middle	47	45
After	36	44
Correlations (%)		
Corr. (length, haircut)	62	91
Corr. (length, partial default)	54	74

We can also compute in our model the haircuts that lenders experience after default episodes, the maturity extensions during episodes, and the correlation between haircuts and episode length. We compare our model statistics with empirical estimates of haircuts and maturity extensions in Josefin Meyer and Trebesch (2018) and in Fang, Schumacher, and Trebesch (2016), and estimates of the comovement of haircuts and default episode length in Benjamin and Wright (2013).[12] These authors find that haircuts after default episodes are 36% on average and that resolutions of default episodes involve debt exchanges with maturity extensions that increase the duration of the debt by 5.5 years. As shown in Table 8, our model also predicts sizable haircuts and maturity extensions during default episodes. Haircuts and maturity extensions in the model are on average 30% and 5.4 years, respectively, which are close to the estimates for the data.

Finally, in terms of correlations, our model predicts that longer default episodes feature larger haircuts and higher levels of partial default, as in the data. The magnitudes of the correlations in the model of 0.91 and 0.74, however, are larger than the data correlations of 0.62 and 0.54.

4.6 Comparative Statics

In this section we perform two main comparative statics in our model that resemble recent discussions around resolution mechanisms for sovereign defaults. The first comparative static considers an environment where the country does not have access to international markets when default is positive. We show that such a comparative static is similar to adding more stringent pari passu clauses to the bond contracts in our model. The second comparative static introduces stochastic discounts during defaults such that with probability π_r, the face value of the bond does not experience a deep reduction. We use this comparative static to study debt reprofiling, a policy supported recently by the International Monetary Fund, which consists of relying more on maturity extensions and less on debt relief for resolving defaults. Our results suggest that pari passu clauses lead to lower default frequency, shorter default episodes, and shorter debt haircuts from defaults. Reprofiling policies reduce haircuts but, in contrast to pari passu clauses, aggravate default episodes by generating higher default frequencies, longer default episodes, and higher debt-to-output ratios.

[12]We use these external sources for these estimates as our dataset does not contain enough information to compute the implied haircuts and maturity extensions given that it lacks the entire cash-flow structure of the debt.

4.6.1 No Market Access during Default and Pari Passu

An important feature of our model that differs from many sovereign default models is that the economy continues to have access to financial markets during default episodes. As we have seen, bond markets are endogenously somewhat restricted in our model because of elevated default risk during periods with positive partial default, but modest levels of new borrowing continue to be issued. In this subsection, we explore the effects of financial market access during defaults by comparing our model to a modified framework in which new borrowing during periods of partial default carries an additional cost that effectively shuts down market access during default.

Recall that in our model, all the coupon payments due from past issuances are combined in a single state variable a. Payments for these coupons are treated equally across issuances as the partial default choice d is applied to the entire sum of coupons a. Nevertheless, empirical measures of haircuts as derived in equation (18) will differ across vintages of bonds issued at different points in time during a default episode. For example, haircuts on bonds issued later in the default episode are smaller than the haircuts for the original legacy debt the economy has at the beginning of the episode. The reason is that bonds issued later in the episode experience fewer periods with positive partial default compared to bonds issued earlier.[13] Such instances of differential treatment across bonds during default episodes are classic violations of pari passu clauses in bond markets, which require that all creditors are treated equally.[14]

In this context, the comparative static we consider in this section that shuts down new borrowing during default episodes ameliorates pari pasu concerns. In fact, we can directly interpret the additional costs for borrowing during default episodes as costs from additional pari passu clauses and potential litigation during default.

In Table 9 we compare the results from an economy with no market access during default to our benchmark model. Shutting down market access during defaults has a sizable impact on the distribution of partial default. Without market access, partial defaults are less frequent and more homogeneous. The default frequency drops from 34% to 14%, and small and large defaults are largely eliminated, reflected by a drop in the standard deviation of partial default by almost half.

[13] Average haircuts in the benchmark model during default episodes of 5 years are, for example, 25% for the legacy debt outstanding at the beginning of the episode and 10% for bond issues during the last year of the episode.

[14] As explained in Olivares-Caminal (2013), bonds with pari passu clauses, sometimes called most favored creditor clauses, stipulate that "during default episodes, if subsequent settlements have better terms, those terms will also be extended to the previously exchanged bonds."

These changes in partial default are reflected in smaller and less volatile interest rate spreads.[15]

In terms of default episodes, no market access during default shortens the length of the episodes by 20%. Default episodes in this experiment shorten for two reasons. First, not having access to international markets adds extra costs from defaults. These higher costs from being in default encourage faster exit from default episodes. Second, debt increases less during default episodes than in the benchmark because no new bond issuances occur. In this comparative static, for example, debt to output increases only about 5% during default episodes that last 10 years, compared to an increase of close to 20% for the benchmark model, as seen in Figure 6. With less debt accumulated during default episodes, the economy can exit default faster. Shorter episodes with no market access also lead to smaller haircuts and somewhat smaller maturity extensions.

This experiment suggests that pari passu clauses can remove market access during default episodes, which could have sizable implications for defaults, making them less frequent, more homogeneous, and shorter.

4.6.2 Stochastic Discounts during Default and Reprofiling

Another important feature of our model that differs from the sovereign default literature is that the defaulted debt does not dissipate after default but accumulates with κ fraction of the defaulted debt due in the future. The fraction $(1-\kappa)$ is the debt discount from default. In this subsection, we explore the implications of making κ stochastic and analyze episodes with smaller discounts. During these episodes, the economy experiences maturity extensions on the defaulted debt and smaller debt relief, a policy akin to reprofiling the debt.

In recent years, the International Monetary Fund has proposed a strategy of reprofiling debt as a resolution mechanism for defaults for countries with intermediate levels of debt sustainability. This strategy consists of limiting restructurings and debt relief while relying on maturity extensions during debt crises for borderline cases (See International Monetary Fund (2015)). The idea behind this strategy is to maintain the average needed restructurings for countries but expand the state contingencies for intermediate cases. We can analyze these policy proposals by making debt discounts stochastic without changing the mean discount and analyzing simulation realizations of

[15] Although the economy is not issuing any new loans during periods of positive partial default in this comparative static, the debt accumulates as defaulted debt during the episode and carries a corresponding interest rate spread priced from secondary markets. The measure of spreads we use for this comparative static includes these secondary market spreads from the accumulated defaulted debt.

Table 9: Comparative Statics

	Benchmark	Pari Passu (No Mkt Access)	Reprofiling (κ Stochastic $\|\kappa = \kappa_H$)
Partial Default $\| > 0$			
Partial default frequency	34	14	47
Partial default mean	31	26	60
Partial default st. dev.	20	11	40
Small partial defaults	7	11	11
Large partial defaults	64	46	100
Default Episodes			
Mean episode length	5	4	8
Fraction of short episodes (≤ 2)	45	45	??
Haircut	30	26	27
Duration extension	5.4	5.3	5.3
Corr. (length, partial default)	74	60	26
Corr. (length, haircut)	91	91	33
Overall First and Second Moments			
Spread mean	1.2	0.3	50
Spread st. dev.	3.9	0.5	115
Debt-to-output mean	36	34	54
Debt-to-output st. dev.	18	14	40
Corr. (partial default, spread)	59	83	69
Corr. (partial default, debt)	72	51	92
Corr. (partial default, output)	-70	-54	-52
Corr. (spread, output)	-32	-57	-31
Corr. (spread, debt)	47	52	81

small discounts.

We implement this idea by assuming that the discount takes two values $\kappa_t = \{\kappa - \varepsilon, \kappa + \varepsilon\}$ that occur with equal probability such that the mean discount is equal to the benchmark economy, with a value for $\varepsilon = 0.5$. Simulation results with a stochastic discount with realizations of κ_t following its transition matrix give average statistics that are almost identical to the ones for the benchmark model. This implies that stochastic discounts by themselves do not change the implications for spreads and default episodes. Results change, however, if we condition on the high or low realization. We interpret simulations with a high κ_t as cases when a reprofiling policy was implemented. In Table 9 we report results for this case. We simulate the model with stochastic discounts by feeding into the economy a sequence of high realizations for $\kappa_t = \kappa + \varepsilon$.

The results from this experiment suggest that with smaller discounts κ_t, partial defaults occur more often and are higher on average. The economy counteracts the smaller discounts by defaulting more to achieve sufficient debt relief. Large partial defaults, for example, are 100%, much higher than the baseline average of 64%. In response to more default, spreads are much higher and volatile in this case. Debt-to-output ratios also increase by about 20% with smaller discounts on the defaulted debt. Default episodes are also longer here as debt grows faster during the default episodes when debt relief is small, requiring a large boom in output for the economy to choose to exit the default. These endogenous responses imply that haircuts and maturity extensions are changed only modestly in equilibrium despite the smaller discounts.

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A Computation

We describe computation aspects for the model with stochastic recovery κ. This encompasses the model in the main part of the paper, and also the extended model used in subsection **??**.

Model with stochastic κ. In this case, κ can take on values κ_i in some discrete set with an iid probability π_{κ_i}. Since this extension does not affect the definition of the state, the model is as set out in section 3 except for a few simple extensions. The evolution of debt payments for the sovereign includes κ_i to become

$$a' = \delta a + (1-\delta)\kappa_i da + b. \tag{A.1}$$

The value to the lender evaluates continuation values over the distribution of κ so that

$$\begin{aligned} H(z,a,y) &= 1 - d(z,a,y) \\ &+ \frac{1}{R}\sum_i \pi_{\kappa_i}\left[\delta + (1-\delta)\kappa_i d(z,a,y)\right] \sum_{z'} \pi(z',z) H(z', a'(z,a,y,\kappa_i), y'(z',z,a,y)) \end{aligned} \tag{A.2}$$

where $a'(z,a,y,\kappa_i) = \delta a + (1-\delta)\kappa_i d(z,a,y)a + b(z,a,y)$.

The expected value conditional on a', that is after the realization of the recovery shock κ_i, gives an ex-post price

$$\tilde{q}(z,a',d) = \frac{1}{R}\ E_z[H(z',a',y')]. \tag{A.3}$$

The relevant price is the ex-ante value over the distribution of κ_i:

$$q(z,b,d,a) = \sum_i \pi_{\kappa_i} \tilde{q}(z, \delta a + b + (1-\delta)\kappa_i da, d).$$

This is now the price showing in the sovereign's constraint which becomes

$$c = y - a(1-d) + q(z,b,d,a)\ b \tag{A.4}$$

Grids and interpolation. We define grids for the state (z,a,y), and the decisions (b,d). Continuation values V and H need to be evaluated on real values for y' and a' off the grid. Also we represent borrowing and default decisions, b and d, that are continuous (at least piece-wise). Therefore, in order to obtain q, $\tilde{q}$ must be evaluated on continuous values a'. Likewise, q will have to be evaluated on continuous values for b and d. To achieve this, we have considered a mixture of interpolation schemes, including linear and cubic splines. Since outcomes are robust, the results reported are based on the following choice: bilinear interpolation over (y, a) for V and H; bilinear interpolation over (b, d) for q; linear interpolation over a' for $\tilde{q}$.

Discontinuity of default. The default decision rule $D(z, a, y)$ may not be continuous everywhere. One reason is that in some states there is a fixed component to the cost of defaulting, i.e., a discrete drop in Ψ as d becomes strictly positive. There, default will jump discontinuously. Treating $d(z, a, y)$ as continuous in (y, a) is inappropriate, and may produce misleading outcomes: Interpolation near the discontinuity may appear to yield some default where there is no default, or some partial default where there should be full default. To deal with this situation, we introduce a discrete decision variable for the extensive margin, whether to default or not at all, $d_{01} \in \{0, 1\}$. The agent's decision is broken down as follows. There is a value to defaulting

$$V^D(z, a, y) = \max_{b,d} \left\{ u(c) + \beta \sum_j \pi_{\kappa_j} \sum_{z'} \pi(z', z) V(z', a', y') \right\}$$

subject to the constraints above. This yields continuous rules $d^D(.)$ and $b^D(.)$. And there is a value to not-defaulting

$$V^{ND}(z, a, y) = \max_{b} \left\{ u(c) + \beta \sum_{z'} \pi(z', z) V(z', \delta a + b, z') \right\},$$

which yields $b^{ND}(.)$. The ex-ante value

$$V(z, a, y) = \max\{V^D(z, A, z), V^{ND}(z, a, y)\}$$

yielding the extensive-margin discrete default decision $d_{01}(z, a, y)$. The enveloping intensive-margin default rule is

$$d(z, a, y) = \begin{cases} 0 & d_{01}(z, a, y) = 0 \\ d^D(z, a, y) & D_{01}(z, a, y) = 1 \end{cases}$$

The discontinuity of default will cause H to be discontinuous where the discrete decision $d_{01}(...)$ switches. Two sections are defined accordingly, one for the default D states and another for no-default ND states. In a defaulting state

$$H^D(z,a,y) = (1 - d(z,a,y)) + \frac{1}{R} \sum_j \pi_{\kappa_j} \left(\delta + (1-\delta)\kappa_j d(z,a,y)\right) \sum_{z'} \pi(z',z) H(z',a',y'),$$

while in a no-defaulting state

$$H^{ND}(z,a,y) = 1 + \frac{1}{R} \delta \sum_{z'} \pi(z',z) H(z',a',y'),$$

where $a' = \delta\ a\ + b(z,a,y)\ +\ (1-\delta)\kappa_j\ d(z,a,y)a$, $y' = z'\ \Psi(d(z,a,y),z')$. So H is:

$$H(z,a,y) = d_{01}(z,a,y) H^D(z,a,y) + (1 - d_{01}(z,a,y)) H^{ND}(z,a,y).$$

Continuation values. In the agent's decision problem, one issue is how to evaluate the continuation values of the envelope function V at points outside of the 2-dim grid defined over (y,a). Such a point would be inside a box defined by the four grid points, say $(y_j, y_{j+1}, a_i, a_{i+1})$. If $d_{01}(z', a_i, y_{j+1}) = 1$, defaulting also must be the option in all the 4 points, and V obtains by interpolating V^D between the four grid points. If $D_{01}(z', a_{i+1}, y_j) = 0$, no-defaulting also must be the option in all the 4 points, and V obtains by interpolating V^{ND} between the four grid points. In the other cases, where the default extensive-margin discrete choice is not the same in all points of the box, interpolate V^D and V^{ND} separately, and pick the value that dominates. For the price function,

$$\tilde{q}(z,a',d) = \frac{1}{R} \ \sum_{z'} \Gamma(z',z) H((z',A',z'+\epsilon')\Psi(d,z'))$$

we have the continuation value of H which has to be evaluated at points outside of the 2-dim grid defined over (y,A). We proceed similarly as above, defining a 2-dimensional box and interpolating either H^D or H^{ND}.

Optimization. In the default case $d_{01} = 1$, over the two dimensions b and d, use the minimization routine PRAXIS based on the principal axis method (Richard Brent; FORTRAN90 version by John Burkardt. URL: `http://people.sc.fsu.edu/~jburkardt/f_src/praxis/praxis.html`). For the no-default case $d_{01} = 0$, over the single dimension b, we use Golden Section search, with the bracketing using Fortran90 intrinsic procedure `maxloc` on the grid. The algorithm for our moment-matching exercise is based on the software BOBYQA, authored by Michael J. D. Powell, that minimizes the sum of squares of the target moments with bound constraints, by combining the trust region method and the Levenberg-Marquardt method.

Algorithm. The solution is found by iterating backwards starting from some terminal conditions. The objects that need initialisation are as follows. The algorithm starts from the outcomes of a finite economy which means the following terminal values: $H = H^D = 0$, $H^{ND} = 1$, $v^D = u(y)$, $v^{ND} < v^D$, and $d_{01} = 1$. The algorithm in each iteration solves the following sequence: updating of debt prices; agent decision; lender values. Specifically, initialise H^D, H^{ND}, H, d_{01}, v^D, v^{ND} and iterate over the following updating sequence:

- Prices

  $H^D, H^{ND}, D_{01} \mapsto \tilde{q}$

  $\tilde{q} \mapsto q$

- Sovereign:
  - $q, v^D, v^{ND}, d_{01} \mapsto v^D, b^D, d^D, d$
  - $q, v^D, v^{ND}, d_{01} \mapsto v^{ND}, b^{ND}$
  - $v^D, v^{ND}, b^D, b^{ND}, d^D \mapsto d_{01}, b, d$

- Prices:
  - $H, b^D, d^D \mapsto H^D$
  - $H, b^{ND} \mapsto H^{ND}$
  - $H^D, H^{ND}, d_{01} \mapsto H$

The algorithm is based on the Software BOBYQA, authored by M. J. D. Powell, to minimize sum of squares with bound constraints by combing trust region method and Levenberg-Marquardt method.

Convergence. Regarding convergence, one clear challenge is the feedback between the agent's decisions d and b and the contract valuation H and therefore prices q. The practical procedure must involve a first long run of iterations without dampening, possibly followed by some dampening. We have found that dampening is not necessary for our main results.

Convexity and multiple local optima. The value function (as a function of debt a) may become eventually flat, so it must feature a convex section. This is because when default approaches

100% the penalty cannot increase further even if debts increase. The convex section can create multiplicity and discontinuous decisions.This may be relevant in equilibrium because, given that debt is long term, there is positive recovery even at high-default positions so debt prices do not fall fast enough to prevent this type of situations. Nonetheless, in our calculations these problematic regions appear to lie outside of the ergodic set and pose no difficulty.

B Haircut and maturity extensions

Remember coupon payments are such that: $a_{t+1} = \delta a_t + b_t + (1-\delta)Rd_t a_t$.

Without default risk. It is useful to start with the case without default risk. The value of the payments to a bond that pays one unit 1 today is:

$$val = \sum_{t=1}^{\infty} \left(\frac{\delta}{1+r}\right)^{t-1} = \frac{1+r}{1+r-\delta}$$

Its duration measured by the Macaulay duration:

$$dur \equiv \frac{1}{val}\sum_{t=1}^{\infty} t\left(\frac{\delta}{1+r}\right)^{t-1} = \frac{\sum_{t=1}^{\infty} t\left(\frac{\delta}{1+r}\right)^{t-1}}{\sum_{t=1}^{\infty}\left(\frac{\delta}{1+r}\right)^{t-1}} = \frac{\frac{1}{(1-\frac{\delta}{1+r})^2}}{\frac{1+r}{1+r-\delta}} = \frac{1+r}{1+r-\delta},$$

where the second equality follows from the fact that $\sum tx^{t-1} = 1/(1-x)^2$.

Old debt instruments. The present value and the duration of the of old defaulted debt instruments DD from time zero are

$$\text{value}(DD) = \sum_{t=1}^{N} \frac{d_t a_t}{(1+r)^{t-1}}, \quad \text{dur}(DD) = \frac{1}{\text{value}(DD)}\sum_{t=1}^{N} t\frac{d_t a_t}{(1+r)^{t-1}}.$$

New debt instruments. Regarding the new debt instruments, in each period t there is new coupons due arising from the defaults occurred in the preceding periods $t = 1, 1, 3, ..., t-1$ within the episode we denote a_t^{ND} for "new debt". In other words,

$$a_t^{ND} \equiv (1-\delta)\kappa[d_{t-1}a_{t-1} + \delta d_{t-2}a_{t-2} + \delta^2 d_{t-3}a_{t-3} + ... + \delta^{t-2}d_1 a_1].$$

Note this can be constructed recursively as $a_t^{ND} = (1-\delta)\kappa d_{t-1}a_{t-1} + \delta a_{t-1}^{ND}$ for $t = 2, ..., N+1$ with $a_1^{ND} = 0$. To calculate the current value of these new bonds at t, we subtract the part of a_t^{ND} that is defaulted at t, so it is $a_t^{ND}(1-d_t)$. The present value is calculated as

$$\begin{aligned} valND &= \sum_{t=2}^{N} \frac{a_t^{ND}(1-d_t)}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N} \sum_{t=1}^{\infty} \frac{\delta^{t-1}}{(1+r)^{t-1}} \\ &= \sum_{t=2}^{N} \frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N} \frac{1+r}{1+r-\delta} \end{aligned}$$

The duration of the new debt instruments is therefore

$$\begin{aligned} durND &= \frac{1}{valND} \left[\sum_{t=2}^{N} \frac{a_t^{ND}(1-d_t)}{(1+r)^{t-1}} t + \frac{a_{N+1}^{ND}}{(1+r)^N} \sum_{t=1}^{\infty} \frac{\delta^{t-1}}{(1+r)^{t-1}} (N+t) \right] \\ &= \frac{1}{valND} \left[\sum_{t=2}^{N} \frac{a_t^{ND}(1-d_t)}{(1+r)^{t-1}} t + \frac{a_{N+1}^{ND}}{(1+r)^N} \left(N \sum_{t=1}^{\infty} \left(\frac{\delta}{1+r} \right)^{t-1} + \sum_{t=1}^{\infty} \left(\frac{\delta}{1+r} \right)^{t-1} t \right) \right] \\ &= \frac{1}{\text{value}(ND)} \sum_{t=2}^{N} t \frac{(1-d_t)a_t^{ND}}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND}}{(1+r)^N} \left(N \frac{1+r}{1+r-\delta} + \frac{1}{(1-\frac{\delta}{1+r})^2} \right) \end{aligned}$$

Haircuts for different intages. Pari pasu. Haircuts on different debt vintages within an episode. Consider an episode with $d_t > 0$ over $t = 1, ..., N$. Denote by $a_t^L(j)$ the coupon due in period $t = 2, ..., N+1$ of new (i.e., not-yet-defaulted) debt whose first payment was due in period j, for $j = 1, ..., N+1$. We can refer to j as the vintage of this new debt. It is a fact that $a_t^L(j) > 0$ only if $t \geq j$. More specifically, we have:

$$a_t^L(j) = \begin{cases} \delta^{t-j} a_j & \text{if } j = 1,\ t = 2, 3, ..., N \\ \delta^{t-j} b_{j-1} & \text{if } j = 2, ..., N+1,\ t = j, j+1, ..., N+1 \end{cases}$$

The value of defaulted debt of vintage $j = 1, ..., N$:

$$vDD^L(j) = \sum_{t=2}^{N} \frac{d_t a_t^L(j)}{(1+r)^{t-1}}.$$

The default on new debt generates new restructured debt corresponding to each vintage, $a_t^{ND,L}(j)$, which can be calculated recursively

$$a_t^{ND,L}(j) = (1-\delta)\kappa d_{t-1} a_{t-1}^{L}(j) + \delta a_{t-1}^{ND,L}(j)$$

for $t = 2, 3, ..., N+1$ and for $j = 1, ..., N+1$, with $a_1^{ND,L} = 0$. The value, after adjusting for current default,

$$vND^L(j) = \sum_{t=2}^{N} \frac{a_t^{ND,L}(j)(1-d_t)}{(1+r)^{t-1}} + \frac{a_{N+1}^{ND,L}(j)}{(1+r)^N} \frac{1+r}{1+r-\delta}$$

Let $a_t^{ND,L} = \sum_j a_t^{ND,L}(j)$. Going back to total arrears a^{ND}, we know $a_t = a_t^{ND} + a_t^{L}$, so

$$a_t^{ND} = (1-\delta)\kappa d_{t-1}(a_{t-1}^{ND} + a_{t-1}^{L}) + \delta a_{t-1}^{ND},$$

which makes clear that the difference between a_t^{ND} and $a_t^{ND,L}$ is that the latter does not include coupons defaulted in previous periods. With $vND^L(j)$ and $vDD^L(j)$ we can calculate haircuts for the different vintages j.

$$
\begin{aligned}
0 &= u_c[Q_b b + Q] + \beta\ E\{V_a'\} \\
0 &= u_c[a + Q_d b] + \beta\ E\{(1-\delta)\kappa a V_a' + z'\Psi_d V_y'\} \\
V_y &= u_c \\
V_a &= u_c\left\{-(1-d) + a\frac{\partial d}{\partial a} + b\left[Q_b\frac{\partial b}{\partial a} + Q_d\frac{\partial d}{\partial a} + Q_a\right] + Q\frac{\partial b}{\partial a}\right\} + \\
&\qquad \left[\delta + (1-\delta)\kappa\ d + \frac{\partial b}{\partial a} + (1-\delta)\kappa\ a\ \frac{\partial d}{\partial a}\right]\ \beta\, E\{V_a'\} + \beta\frac{\partial d}{\partial a}\Psi_d\, E\{V_y'\} \\
&= u_c\left\{-(1-d) + bQ_a\right\} + \\
&\qquad [\delta + (1-\delta)\kappa\ d]\ \ \beta\, E\{V_a'\}
\end{aligned}
$$

where the 2nd equality follows from the two FOC's. Using again the 1st FOC to substitute V_a':

$$
V_a = u_c\left\{-(1-d) + bQ_a - (\delta + (1-\delta)\kappa)d)(Q_b b + Q)\right\}.
$$

Use the fact that $Q_a = (\delta + (1-\delta)\kappa)Q_b$, so the terms with Q_a and Q_b drop to get

$$
V_a = u_c\left\{-(1-d) - (\delta + (1-\delta)\kappa)dQ\right\}.
$$